Lecture 6:

Recap:

Theorem: (Rank-Nullity Theorem)

Let V and W be vector spaces s.t. V is finite-dimensional.
Then for any linear transformation $T: V \to W$, we have:

$$\text{nullity}(T) + \text{Rank}(T) = \dim(V)$$

Thm: Let V and W be vector spaces of equal finite-dimensions. Let $T: V \to W$ be a linear transformation.
Then, the following are equivalent:

(a) T is one-to-one

(b) T is onto

(c) $\text{Rank}(T) = \dim(V)$

$\dim(R(T)) \leq \dim(W)$

Proof: T is one-to-one

$\iff \text{Nullity}(T) = 0$ (by previous proposition)

$\iff \text{Rank}(T) + \cancelto{0}{\text{Nullity}(T)} = \dim(V)$

$\iff \underset{\dim(R(T))}{\underset{\parallel}{\text{Rank}(T)}} = \dim(W) \iff R(T) = W$

$\iff T$ is onto

Example: Consider $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by:

$$T(f(x)) \overset{\text{def}}{:=} \underline{2f'(x)} + \underline{\int_0^x 3f(t)\,dt}$$

We have $R(T) = \operatorname{span}\{T(1), T(x), T(x^2)\}$

$$= \operatorname{span}\underbrace{\{3x,\ 2+\tfrac{3}{2}x^2,\ 4x+x^3\}}_{\text{Linear independent}}$$

$$\dim(R(T)) = \operatorname{rank}(T) = 3$$

$$\cancelto{3}{\operatorname{Rank}(T)} + \operatorname{Nullity}(T) = \dim(\cancelto{3}{P_2(\mathbb{R})})$$

$$\Rightarrow \operatorname{Nullity}(T) = 0 \Rightarrow N(T) = \{\vec{0}\}$$

$\Rightarrow$ T is one-to-one.

Example: Show that $\forall q(x) \in P(\mathbb{R})$, $\exists p(x) \in P(\mathbb{R})$ such that

$\forall$ ← for all $\exists$ ← there exists

$$[(x^2+5x+7)p(x)]'' = q(x).$$

(T is onto)

Consider $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by:

$$T(p(x)) = [(x^2+5x+7)p(x)]''$$

(Exercise: T is linear)

(~~Need to check $N(T) = \{\vec{0}\}$ or $\text{Nullity}(T) = 0$~~)

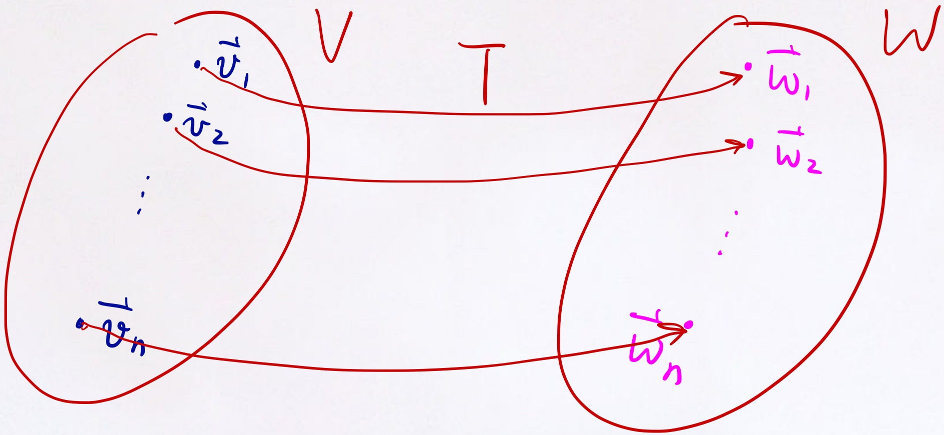
because $\dim(P(\mathbb{R})) = \infty$

Idea: Restrict T to $P_n(\mathbb{R})$: Define: $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

such that $T(p(x)) = [(x^2+5x+7)p(x)]''$

Remain to show $\text{Nullity}(T) = 0$. (Exercise)

Thm: Let V and W be vector spaces. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. Then, given any $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$. $\exists$ a unique linear transformation $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$



Proof: For $\vec{x} \in V$, $\exists !\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$.

We define $T: V \to W$ by: $T(\vec{x}) = \sum_{i=1}^{n} a_i \vec{w}_i \in W$

- T is linear: For $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, $\vec{y} = \sum_{i=1}^{n} b_i \vec{v}_i \in V$

and $c \in F$,

we have:

$$\begin{aligned}
T(c\vec{x} + \vec{y}) &= T\left(\sum_{i=1}^{n} (c a_i + b_i) \vec{v}_i \right) \\
&= \sum_{i=1}^{n} (c a_i + b_i) \vec{w}_i \\
&= c \underbrace{\left(\sum_{i=1}^{n} a_i \vec{w}_i \right)}_{= T(\vec{x})} + \underbrace{\left(\sum_{i=1}^{n} b_i \vec{w}_i \right)}_{= T(\vec{y})}
\end{aligned}$$

- By definition, $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$

- T is unique: Suppose $U: V \to W$ is linear s.t.

$$U(\vec{v}_i) = \vec{w}_i \quad \text{for } \forall i.$$

For any $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, we have:

$$U(\vec{x}) = \sum_{i=1}^{n} a_i \underbrace{U(\vec{v}_i)}_{=\vec{w}_i} = \sum_{i=1}^{n} a_i \vec{w}_i = T(\vec{x}).$$

$$\therefore \quad U = T.$$

Corollary: Let V be a vector space with a finite basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

Then any linear transformation from V to another vector space W is completely determined by its values on β.

(That is, if U and T are linear transformations from V to W s.t. $U(\vec{v}_i) = T(\vec{v}_i)$, then $U = T$)

Matrix representation

Notation: An ordered basis for a finite-dimensional vector space V is a basis for V endowed with a specific order.

(e.g. $\mathbb{R}^2$ $\underbrace{\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}}_{=\beta_1} \neq \underbrace{\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}}_{=\beta_2}$ as ordered basis)

Definition: Let V be a finite-dimensional vector space and $\beta = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an ordered basis for V.

Then, $\forall \vec{x} \in V$, $\exists!\ a_1, a_2, \dots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i$.

The coordinate vector of $\vec{x}$ relative to β, denoted as $[\vec{x}]_\beta$, is the column vector (F^n)

$$[\vec{x}]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n$$

Remark: Define a map $V \to F^n$. This map is linear
$$\vec{x} \mapsto [\vec{x}]_\beta$$
(HW. $[a\underbrace{\vec{x}+\vec{y}}_{\in V}]_\beta = a[\vec{x}]_\beta + [\vec{y}]_\beta$)

Now, suppose V and W are finite-dimensional vector spaces with ordered bases $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ (for V) and $\gamma = \{\vec{w}_1, \ldots, \vec{w}_m\}$ (for W) respectively.

Let $T: V \to W$ be a linear transformation.

Then for each $1 \le j \le n$, $\exists\, a_{ij} \in F$ $1 \le i \le m$ such that

$$\underbrace{T(\vec{v}_j)}_{\in W} = \sum_{i=1}^{m} a_{ij} \vec{w}_i \quad \text{for} \quad 1 \le j \le n.$$

Definition: With this notation as above, we call the matrix

$A :\overset{\text{def}}{=} (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ the matrix representation of T in the ordered bases β and γ, and denoted it as $A = [T]_\beta^\gamma$.

$$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij}\,\vec{w}_i \quad \text{for} \quad 1 \le j \le n,$$

$$\in W$$

$$A = \left(\underset{\overset{\shortparallel}{[T(\vec{v}_1)]_\gamma}}{\boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{matrix}}} \quad \underset{\overset{\shortparallel}{[T(\vec{v}_2)]_\gamma}}{\boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{matrix}}} \quad \cdots \quad \underset{\overset{\shortparallel}{[T(\vec{v}_n)]_\gamma}}{\boxed{\begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix}}} \right)$$

$$T(\vec{v}_1) = \sum_{i=1}^{m} a_{i1}\,\vec{w}_i$$

$$\in W$$

$$\Downarrow$$

$$[T(\vec{v}_1)]_\gamma = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \in F^n$$

$\beta = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for V

$\gamma = \{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_m\}$ for W

$$[T]_\beta^\gamma = \begin{array}{cc} m & \underbrace{\begin{pmatrix} | & | & & | \\ [T(\vec{v}_1)]_\gamma & [T(\vec{v}_2)]_\gamma & \cdots & [T(\vec{v}_n)]_\gamma \\ | & | & & | \end{pmatrix}}_{n} \end{array}$$

$\in$

$M_{m \times n}$